Sum Labeling for Arbitrary Supersubdivision of Comb, \( P_n \odot 2K_1 \) & Spider

- Gerard Rozario J* - Jon Arockiaraj J**

Abstract

A sum labeling is a mapping from the vertices of \( G \) into the positive integers such that, for any two vertices \( u, v \in V(G) \) with labels \( (u) \) and \( (v) \), respectively, \( (uv) \) is an edge iff \( (u) + (v) \) is the label of another vertex in \( V(G) \). Any graph supporting such a labeling is called a sum graph. It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as isolates and the labeling scheme that requires the fewest isolates is termed optimal. The number of isolates required for a graph to support a sum labeling is known as the sum number of the graph.

In this paper, we will obtain optimal sum labeling scheme for arbitrary super subdivision of comb, and spider

**Keywords**: Sum Labeling, Sum Number, Sum Graph, Path Union, Arbitrary Supersubdivision

2010 AMS Subject Classification: 05C78

Introduction

All the graphs considered here are simple, finite and undirected. For all terminologies and notations we follow Harary [5] and graph labeling as in [2]. Sum labeling of graphs was introduced by Harary [6] in 1990. Following definitions are useful for the present study.

**Definition 1.1** A sum labeling is a mapping from the vertices of \( G \) into the positive integers such that, for any two vertices \( u, v \in V(G) \) with labels \( (u) \) and \( (v) \), respectively, \( (uv) \) is an edge iff \( (u) + (v) \) is the label of another vertex in \( V(G) \). Any graph supporting such a labeling is called a sum graph.

**Definition 1.2** It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as isolates and the labeling scheme that requires the fewest isolates is termed optimal.

**Definition 1.3** The number of isolates required for a graph \( G \) to support a sum labeling is known as the sum number of the graph. It is denoted as.

**Definition 1.4** [8] Let \( G \) be a graph with \( q \) edges. A graph \( H \) is called a super subdivision of \( G \) if \( H \) is obtained from \( G \) by replacing every edge \( e_i \) of \( G \) by a complete bipartite graph \( K_{2,m_i} \) for some \( m_i, 1 \leq i \leq q \) in such a way that the end vertices of each \( e_i \) are identified with the two vertices of 2-vertices part of \( K_{2,m_i} \) after removing the edge \( e_i \) from graph \( G \). If \( m_i \) is varying arbitrarily for each edge \( e_i \), then super subdivision is called arbitrary super subdivision of \( G \).

**Definition 1.5** The graph obtained by adding a pendent edge to each vertex of a path of \( n \) vertices is called a Comb. It is denoted by.

**Definition 1.6** is the graph obtained by adding two pendent edges to each vertex of a path of \( n \) vertices.

*PG and Research Department of Mathematics, St. Joseph's College of Arts and Science, Cuddalore, Tamilnadu, India. rozario.gerard@yahoo.com

**PG and Research Department of Mathematics, St. Joseph's College of Arts and Science, Cuddalore, Tamilnadu, India. rozario.jonarockiaraj@gmail.com
Definition 1.7 (Chung et. al [1]) A tree is called a spider if it has a center vertex c of degree $k > 1$ and each other vertex either is a leaf (pendent vertex) or has degree 2. Thus a spider is an amalgamation of $k$ paths with various lengths. If it has $x_1$ paths of length $a_1$, $x_2$ paths of length $a_2$, ..., $x_n$ paths of length $a_m$, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, ..., a_m^{x_n})$ where $a_1 < a_2 < ... < a_m$ and $x_1 + x_2 + ... + x_n = k$.

Optimal Sum Labeling Scheme for Arbitrary Super Subdivision of Comb, & Spider

Sethuraman et.al [8], introduced a new method of construction called Supersubdivision of graph and proved that arbitrary supersubdivision of any path and cycle $C_n$ are graceful. Kathiresan et.al [7], proved that arbitrary supersubdivision of any star is graceful. In [3], Gerard Rozario et.al proved that arbitrary super subdivision of path, cycle and star are sum graph with sum number 2. In [4], Gerard Rozario et.al proved that arbitrary super subdivision of crown, armed crown and t-thorny ring are sum graph with sum number 2.

In this section, we prove that graphs obtained by arbitrary super subdivision of comb, and spider are sum graph with sum number 2.

**Theorem: 2.1**

Arbitrary supersubdivision of a comb is sum graph with sum number 2.

**Proof:** Let $G$ be a comb. Let $v_i (1 \leq i \leq n)$ be the vertices of path and $w_i$ be the pendent vertex adjacent to $v_i (1 \leq i \leq n)$. Let $H$ be the arbitrary supersubdivision of $G$ which is obtained by replacing every edge of $G$ with $K_{2,m}$. Let $u_j$ be the vertices which are used for arbitrary supersubdivision of $G$ where $1 \leq j \leq m$. Let $x$ and $y$ be two isolated vertices. Therefore, the vertex set of $H$ is $V(H) = \{ v_1, v_2, ..., v_n, w_1, w_2, ..., w_n, u_1, u_2, ..., u_m\}$. Define $f : V(H) \rightarrow \mathbb{N}$

- $f(v_1) = 1$
- $f(v_i) = f(v_{i-1}) + 2$ for $2 \leq i \leq n$
- $f(w_1) = f(w_{i-1}) + 2$ for $2 \leq i \leq n$
- $f(u_1) = m + n$
- $f(u_j) = f(u_{j-1}) - 1$ for $2 \leq j \leq m$
- $f(x) = f(u_1) + 1$ and $f(y) = f(u_2) + 2$

Thus, arbitrary supersubdivision of comb is sum graph with sum number 2.

**Illustration:** Sum labeling for arbitrary supersubdivision of comb $P_4 \bigcirc K_1$ is shown in figure 2.1

**Theorem: 2.2** Arbitrary supersubdivision of $P_n \bigcirc 2K_1$ is sum graph with $\sigma(G) = 2$.

**Proof:** Let $G$ be the graph $P_n \bigcirc 2K_1$. Let $p_i (1 \leq i \leq n)$ be the vertices of path and $p_{i1}$ and $p_{i2}$ be the pendent vertices adjacent to $p_i (1 \leq i \leq n)$. $G$ has $3n$ vertices and $(3n - 1)$ edges. Let $H$ be the arbitrary supersubdivision of $G$ which is obtained by replacing every edge of $G$ with $K_{2,m}$. Let $m = \sum_{i=1}^{3n-1} m_i$. Let $u_i$ be the vertices which are used for arbitrary supersubdivision of $G$ where $1 \leq i \leq m$. Let $x$ and $y$ be two isolated vertices. Therefore, the vertex set of $H$ is $V(H) = \{ p_1, p_2, ..., p_n, p_{i1}, p_{i2}, ..., p_{i,m}, u_1, u_2, ..., u_m\}$. Define $f : V(H) \rightarrow \mathbb{N}$

- $f(p_1) = 1$; $f(p_{i1}) = 2$; $f(p_{i2}) = 3$
- $f(p_i) = f(p_{i-1}) + 3$
- $f(p_{i1}) = f(p_{i-11}) + 3$
- $f(p_{i2}) = f(p_{i-12}) + 3$
- $f(u_1) = 3n + m$
- $f(u_j) = f(u_{j-1}) - 1$ for $2 \leq j \leq m$
- $f(x) = f(u_2) + 1$ and $f(y) = f(u_2) + 2$

Thus, arbitrary supersubdivision of $P_n \bigcirc 2K_1$ is sum graph with sum number 2.

**Illustration:** Sum labeling for arbitrary supersubdivision of $P_4 \bigcirc 2K_1$ is given in figure 2.2

**Figure 2.1**

**Figure 2.2**
**Theorem:** 2.3 Arbitrary supersubdivision of the spider $SP_n$, is sum graph with $\sigma(G) = 2$.

**Proof:** Let $G = SP_n$, where $SP_n$ is a spider of $n$ vertices. $G$ has $n$ vertices and $n-1$ edges. Let the vertices of $SP_n$ be $v_1, v_2, \ldots, v_n$. Let $H$ be the arbitrary supersubdivision of $G$ which is obtained by replacing every edge of $G$ with $K_{2,m_i}$. Let $m = \sum_{i=1}^{n-1} m_i$. Let $u_i$ be the vertices which are used for arbitrary supersubdivision of $G$ where $1 \leq i \leq m$. Let $x$ and $y$ be two isolated vertices. Therefore, the vertex set of $H$ is $V(H) = \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m \}$.

Define $f : V(H) \rightarrow \mathbb{N}$

- $f(v_i) = i$ for $1 \leq i \leq n$
- $f(u_i) = n + m$
- $f(u_i) f(u_{i+1}) + 1$ for $2 \leq i \leq m$
- $f(x) = f(u_1) + 1$ and $f(y) = f(u_1) + 2$

Thus, arbitrary supersubdivision of spider $SP_n$, is sum graph with sum number 2.

**Illustration:** Sum labeling for arbitrary supersubdivision of spider is shown in figure 2.3

![Figure 2.3](image)

**Reference**

5. Harary F (1972), *Graph theory*, Addison Wesley, Reading, Massachusetts.