Department of Mathematics

QUESTION BANK

Class: III B.Sc Mathematics

Sub Name: REAL ANALYSIS-II

Sub Code: MT615

Staff Name: Mrs.C.Devi Shyamala mary

UNIT-I

2-Marks

1. Define a subset A of M is bounded.
2. Define a subset A of M is totally bounded.
3. Give an example of a set which is bounded but not totally bounded.
4. Define connected set.
5. Given an example for a subset of R^1 which is not connected.
6. Define connected set with example.
7. Give an example of connected set in R^1.
8. A=[0,1]U[2,3]. Is the set is connected. Give the reason to your answer.

5-Marks

1. If A_1 and A_2 are connected subsets of a metric space M, and if $A_1 \cap A_2 \neq \phi$ then show that $A_1 \mathrm{U} A_2$ is also connected.
2. Let f be a continuous function from a metric space M_1 into a metric space M_2. If M_1 is connected, then prove that the range of f is also connected.
3. Prove that the interval [0,1] is not a connected subset of R_d.
4. Prove that the subset A of R' is connected iff whenever a$\in A$,b$\in A$ with a<b, then c$\in A$ for any c such that a<c<b.

10-Marks

1. Prove that the subset A of R' is connected if and only if whenever $a \in A$, $b \in A$ with $a < b$ then $c \in A$ for any c such that $a < c < b$.
2. If the subset A of the metric space $\langle M, \rho \rangle$ is totally bounded, then prove that A is bounded.
3. Let M be a metric space .Prove that the subset A of M is totally bounded if and only if every sequence of points of A contains a Cauchy subsequence.
4. Let M be a metric space and let A be a subset of M. Then if A has either one of the following properties it has the other.
   i)It is impossible to find non-empty subsets A_1, A_2 of M such that A=$A_1 \cup A_2$,$\overline{A_1} \cap A_2 = \emptyset$ and $A_1 \cap \overline{A_2}$=$\emptyset$.
   ii) When $\langle A, \rho \rangle$ it itself regarded as a metric space, then there is no set except A and $\emptyset$ which is both open and closed in $\langle A, \rho \rangle$.

UNIT-II

2-Marks

1. Define complete metric space with example.
2. Define contraction map on a metric space $\langle M, \rho \rangle$.
3. State the Picard fixed-point theorem.
4. State finite intersection property.
5. Define a compact metric space and give an example of a space which is not compact.
6. Define inverse image.
7. Define continuous at a point p.

5-Marks

1. State and prove generalization of the nested interval theorem in a complete metric space.
2. Show that a closed subset of a complete metric space is complete.
3. If A is a closed subset of the compact metric space $\langle M, \rho \rangle$, then the metric space $\langle A, \ \rho \rangle$ is also compact.
4. Show that a closed subset of a compact metric space is compact.
5. Let $\langle M, \rho \rangle$be a complex metric space for each n ∈I ,letF_n be a closed bounded subset of M such that ,i)$F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$ and ii)diam $F_n \to 0$ as n→ ∞ ,then show that $\underset{n=1}{\overset{\infty}{\cap}} F_n$ contains precisely one point.
6. Prove that if the real-valued function *f* is continuous on the compact metric space *M*, then *f* attains a maximum value at some point of *M*. Also, *f* attains a minimum value at some point of *M*.
7. If the real-valued function f is continuous on the closed bounded interval [a,b]then prove that attains a maximum and a minimum value at points of [a, b].

10-Marks

1. Prove that, the metric space <M ,p> is compact if and only if every sequence of points in M has a subsequence converging to a point in M.
2. Prove that a contraction *f* of a complete metric space *S* has a unique fixed point *p*.
3. State and prove Picard's fixed point theorem.
4. If the metric space M has the Heine-Borel property then prove that M is compact.
5. Let$\langle M, \rho \rangle$ be a complete metric space. If *T* is a contraction on *M* , then show that there is one and only one point *x* in *M* such that $Tx = x$.

6. If M is a compact metric space, then M has the Heine-Borel property .

UNIT-III

<u>2-Marks</u>

1. If each of the subsets E1 ,E2,... of R^1 is of measure ,then $\bigcup_{n=1}^{\infty}$ En is also of measure Zero.
2. If $f \in [a,b]$, g∈R$[a,b]$ and if f(x)≤g(x) almost everywhere (a≤x≤b), then $\int_a^b f \le \int_a^b g$.
3. Define derivatives.
4. Define the Riemann integrable on$[a,b]$.
5. State the Rolle's Theorem.
6. For each $n \in I$ let σ_n be the subdivision $\left\{0,\frac{1}{n},\frac{2}{n} \ldots .\frac{n}{n}\right\}$ of [0,1].compute lim U[f; σ_n] as n→ ∞
7. Show that the function *f* $(x) = x^2$is derivable on [0,1].
8. Prove that every countable subset of R^1 has measure zero.
9. What is mean by measure zero.
10. Find a suitable point of c of Rolle's theorem for f(x)=(x-a)(x-b), (a$\le x \le b$).

<u>5-Marks</u>

1. Let D be a bounded function on [a,b]. Then every upper sum for j is greater than or equal to every lower sum for f. That is, if σ and τ are any two subdivisions of [a ,b], then U[f;σ]$\ge L[f;\tau]$.
2. If $f \in R[a,b]$ then |f|$\in R[a,b]$ and$|\int_a^b f| \le \int_a^b |f|$.
3. Prove that if $f \in [a,b]$and $a < c < b$, then $f \in [a,c], f \in [c,d]$ and$\underline{\int_a^b f = \int_a^c f + \int_c^b f}$.
4. If f∈R$[a,b]$,g∈R$[a,b]$, then show that f+g∈R$[a,b]$ and $\underline{\int_a^b (f+g) = \int_a^b f + \int_a^b g}$.
5. Prove that if,$f \in [a,b]$ and λ is any real number, then and $\underline{\int_a^b \lambda f = \lambda \int_a^b f}$.

6. If the real-valued function f has a derivative at the point $c \in R^1$, then prove that f is continuous at c.
7. If f has a derivative at every point of $[a,b]$, then show that f' takes on every value between $f'(a)$ and $f'(b)$.
8. State and prove Rolle's theorem.
9. Let f be continuous real valued function on the closed bounded interval [a ,b]. If the maximum value for f is attained at c where a<c<b, and if f'(c) exists, then f'(c)=0.

10-Marks

1. Let f be a bounded function on [a, d],then $f \in [a, b]$ if and only if for each $\epsilon > 0$ there exists a subdivision σ of [a, b] such that U$[f;\sigma] < L[f;\sigma] + \epsilon$.

2. If f∈R$[a, b]$,g∈R$[a, b]$, then show that f+g∈R$[a, b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
3. Prove that, if $f \in [a, b]$ and $a < c < b$, then $f \in [a, c], f \in [c, d]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.
4. Prove that if,$f \in [a, b]$ and λ is any real number, then and $\int_a^b \lambda f = \lambda \int_a^b f$.
5. If f be a 1-1 real valued function on an interval J.Let φ be the inverse function for f. If f is continuous at $c \in J$, and if φ has a derivative at d=f(c) with $\varphi'(d) \neq 0$,then show that $f'(c)$exists and $f'(c) = \frac{1}{\varphi'(d)}$.

6. State and prove the Rolle's Theorem.
7. Suppose f has a derivative at c and g has a derivative at f(c).then $\varphi = g.f$ has a derivative at c and $\varphi'(c) = g'[f(c)]f'(c)$.

UNIT-IV

2-Marks

1. State the Law of the Mean.
2. Prove that the improper integral $\int_1^\infty \frac{1}{x} dx$is diverges.
3. Define Cauchy principle value.
4. Examine the convergence of$\int_0^1 \frac{1}{x^2} dx$.
5. If f(x)=0 for every x in the closed bounded interval [a, b] then f is constant on [a, b].
6. State the important application of the law of mean.
7. If f'(x)=g'(x) for all x in the closed bounded interval [a ,b] then f-g is constant(ie) ,f(x)=g(x)+c $(a \le x \le b)$ for some c∈R.

5-Marks

1. Prove that if $f'(x) = 0$ for every x in the closed bounded interval $[a,b]$, then f is constant on $[a,b]$, i.e., $f(x) = C$ $(a \le x \le b)$ form some $C \in R$.
2. State and prove the Law of the Mean.
3. Let f and g be continuous function on the closed bounded interval [a ,b] with g(a) $\neq$ g(b) . If both f and g have a derivative at each point of (a ,b), and f'(t)&g'(t) are not both equal to zero for any t$\in$(a ,b) , then prove that there exist a point C $\in$ (a ,b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

4. Prove that the improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$ is divergent.
5. State and Prove the First fundamental theorem of calculus.
6. If f is continuous function on the closed bounded interval [a, b], and if $\varphi'(x) = f(x)(a \le x \le b)$,then $\int_a^b f(x)dx = \varphi(b) - \varphi(a)$.

10-Marks

1. State and prove the second fundamental theorem of calculus.
2. Prove that the improper integral $\int_0^1 \frac{1}{x} dx$ diverges.
3. Prove that improper integrals $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ converges.
4. State and prove first mean value theorem.
5. If $f \in [a,b]$, if $F(x) = \int_a^x f(t)dt \quad (a \le x \le b)$, and if f is continuous at $x_0 \in [a,b]$,then prove that $F'(x)$=f(x_0).
6. If f is continuous real valued function on the interval J and if $f'(x) > 0$ for all x in J except possibly the end points of J, then prove that f is strictly increasing on J.

UNIT-V

2-Marks

1. State the Taylor's formula with the integral form of remainder.
2. Evaluate lim $\left[\frac{\log x}{x}\right]$ as $x \to \infty$.
3. State Bionomial theorem.
4. State Binomial series.
5. Write the Taylor's expansion of a function f(x) at x=a.
6. Evaluate $\lim_{x\to 0^+} \left[\frac{\tan x - x}{x - \sin x}\right]$.
7. Find $\lim_{x\to 0} \frac{\sin x}{x\cos x + 2\sin x}$.
8. Write Taylor's series about x=a.
9. Sate the theorem whichestablish Taylor's formula with the Cauchy form of the remainder.

5-Marks

1. Evaluate $\lim_{x\to a}\frac{tanx-x}{x-sinx}$.
2. State and prove the Taylor's formula with the Lagrange form of the remainder.
3. Let f be real valued function on the interval [a , a+h] such that $f^{(n+1)}$ (x) exists for every x∈ [a , a+h] and $f^{(n+1)}$ is continuous on [a ,a+h].Let R_{k+1}(x) $=\frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}$(t) dt (x ∈[a , a+h]; k = 0,1 ,…n),then prove that R_k(x) – $R_{(k+1)}$(x) = $\frac{f^{(k)}(a)}{k!}(x-a)^k$ (x∈[a , a+h]; k= 1,2,…n) .
4. If $f(x) = e^x$ then $f'''(x) = e^x$ with n=2 and $e^{0.1} \approx 1.105$.prove that $e^{0.1} < 0.0002$.
5. Evaluate $\lim_{x\to 0}\frac{x-tanx}{x^3}$ by using L` Hospital rule.
6. Let ∅ be a continuous function on the closed bounded interval (a,b), and let g be a continuous function on (a,b) such that g(t)≥0,(a≤t≤b).Then prove that there exists a number c with a≤c≤b such that $\int_a^b \emptyset(t)\, g(t)dt = \emptyset(c)\int_a^b g(t)dt$.

10-Marks

1. State and prove Taylor's formula with the Lagrange form of remainder.
2. Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after *n* times approaches the limit $\frac{1}{(n+1)}$ as h approaches zero provided that $f^{n+1}(x)$ is continuous and different from at x=a.
3. State and prove Taylor's theorem
4. State and prove the binomial theorem.
5. Find the series expansion of $f(x) = cosx$ by Maclaurin's theorem.
6. Let f be a real valued function on [a, a+h] such that $f^{(n+1)}$ *is* continuous on [a, a+h] then prove that $f(x) = f(a) + \left(\frac{f'(a)}{1!}\right)(x-a) + (\frac{f^{('')}(a)}{2!})(x-a)^2 + \cdots + (\frac{f^{(n)}(a)}{n!})(x-a)^n + R_{n+1}(x),$ where $R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$.
7. If $m \in R$ is not a negative integer, then $(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n + \cdots$, Provides that $|x| < 1$.