St. Joseph's College of Arts and Science, Cuddalore.
Question Bank
PG Research Department of Mathematics

Class: II M.Sc Mathematics
Subject Name: Complex Analysis II
Subject Code: PMT101 6
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COMPLEX ANALYSIS-II (PMT1016)

UNIT-I

✓ *TWO MARKS:*

1. State weirstrass theorem.
2. Find the Genus of $\cos\sqrt{2}$.
3. Define Entire Function.
4. State Miltage Lefler theorem.
5. State Hurwitz theorem.
6. State Taylor series.
7. State Laurent series.
8. DefinePartial fractions.
9. DefineInfinite products.
10. Define Canonical products.
11. Define Gamma function.

✓ *FIVE MARKS:*

1. State and Prove Hurwitz theorem.
2. State and Prove weirstrass theorem.
3. Obtain the Expansion of the function $\frac{z-1}{z^2}$ as a Taylor's series in power of $(z-a)$ and give the region of validity.
4. Prove that the infinite product $\pi_1^\infty(1+a_n)$ with $1+a_n \neq 0$ converges simultaneously with the series $\sum_1^\infty \log(1+a_n).$
5. State and Prove Taylor series.

✓ *TEN MARKS:*

1. State and Prove Weirstrass theorem.
2. State and Prove Hurwitz theorem.
3. State and Prove Taylor's series.
4. State and Prove Laurent's series.
5. Prove that the infinite product with converges simultaneously with the series $\sum_1^\infty \log(1+a_n).$ whose terms represent the values of the principle branch of logarithm.
6. State and Prove Mitlag-Loffer's theorem.

7. Every function which is meromorphic in the whole plane is the quotient of two entire functions.

8. A necessary and sufficient conditions for the absolute convergence of the product $\pi_1^{\infty}(1+a_n)$ is the convergence of the series $\Sigma_1^{\infty}|a_n|$.

9.There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form $f(z) = z^m e^{g(z)} \pi_1^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+........+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$ where the product is taken over all $a_n \neq 0$ the m_n are certain integers and g(z) is an entire function.

10. Derive Legendre's Duplication Formula.

UNIT-II

- ✓ ***TWO MARKS:***

1. State the Riemann zeta function.
2. State the Jenson formula.

3. What is a functional equation and give an example.
4. Define entire function and give example.
5. State the formula for the number of zeros of the zeta function with $0 \le t \le T$.

✓ *FIVE MARKS:*

1. Prove that $\zeta(s) = 2^s \pi^{(s-1)} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$.
2. State and prove Jensen's formula.
3. Prove that function $\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$
4. Prove that zeta function can be extended to a meromorphic function in the whole plane whose only pole is a simple pole at s=1 with the residue 1.
5. For $\sigma = \operatorname{Re} s > 1$, Prove that $\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} (1 - p_m^{-s})$.
6. If g(z) is a polynomial of degree ≤ h, then prove that the order of $e^{g(z)} \le h$

✓ *TEN MARKS:*

1. State and prove Jensen's formula.
2. State and prove Poisson Jensen's formula.
3. Prove that for $\sigma = \operatorname{Re} s > 1,\ \frac{1}{\zeta(s)} = \prod_{n=1}^{\infty}\left(1 - p_n^{-s}\right)$.
4. Prove that the genus and the order of an entire function satisfy the double inequality $h \le \lambda \le h+1$.
5. Prove that for $\sigma > 1, \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_c \frac{(-z)^{(s-1)}}{e^z - 1}dz$ where $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(1-z)}$ with $-\pi < Im \log(-z) < \pi$.
6. Prove that function $\xi(s) = \frac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$
7. State and prove Hadamard's theorem.

Prove that $\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$.

UNIT-III

✓ *TWO MARKS:*

1. Define Equicontinuous family of function.
2. Define Normal family.

✓ *FIVE MARKS:*

1.If a sequence of meromorphic functions converges in the sense of spherical distance, uniformly on every compact set. Provethat the limit function is meromorphic or identically equal to ∞.Also if a sequence of analytic functions converges in the same sense, Prove that the limit function is either analytic or identically equal to ∞.

2. Let f be a topological mapping of a region Ω onto a region Ω'.If a sequence $\{z_n\}$ or z(t) tends to the boundary of, then prove that $\{f(z_n)\}$ or f(z(t)) tends to the boundary of Ω'.

3. Prove that a locally bounded family of analytic function has locally bounded derivatives.

4. Prove that the family F is totally bounded iff to every compact set $E \subset \Omega$ and every it is possible to find $f_1, f_{2.}, ...f_n \in F$ such that every $f \in F$ satisfies $d(f, f_i) < \in$ on E for some $f_{i.}$

✓ ***TEN MARKS:***

1. Prove that a family of analytic or meromorohic functions if it is normal in the classical sense iff the expressions $\rho(f) = \frac{2|f^{'}(z)|}{1+|f(z)|^2}$ are locally bounded.

2. State and prove Riemann mapping theorem.

3. State and prove Arzela's theorem.

4. Prove that a family $\mathcal{F}$ of analytic functions is normal with respect to F iff the function in

$\mathcal{F}$ are uniformly bounded on every compact set.

5. Prove that a family $\mathcal{F}$ is normal iff its closure with respect to the distance function $\rho(f,g)=\sum_{k=1}^{\infty}\delta_k(f,g)2^{-k}$ is compact.

UNIT-IV

✓ *TWO MARKS:*

1. When will you say that the real valued continuous function satisfies the mean value property.
2. Define Schwarz triangle function.

3. When do you say that a function u(2) satisfies the mean value property.
4. State Harnack's principle.

✓ *FIVE MARKS:*

1. State and prove the mean value property satisfied by harmonic functions.

2. Show that $F(w)=\int_0^w (1-w^n)^{-2/n} dw$ maps $|w|=1$ onto the interior of a regular polygon with n sides.
3. Show that a continuous function u(z) which satisfies the condition $u(z_0)=\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta$ is necessarily harmonic.

✓ ***TEN MARKS:***

1. State and prove the Schwarz- Christoffel formula.
2. Derive Harnack's inequality.
3. State and prove Harnack's principle
4. Prove that the function Z=F(w) which map $|w|$<1 conformally onto polygons with angles $\alpha_k \pi (k=1,4,\ldots\ldots,n)$ are of the form F(w) = C.$\int_0^w \pi_{k=1}^{n} (w-w_k)^{-\beta_k} dw \neq 9c'$ where $\beta_k = 1-\alpha_k$, then w_k are points on the unit circle, and c,C are complex constants.

UNIT-V

✓ ***TWO MARKS:***

1. What is a period module.
2. Define simply periodic function.

3. Define discrete module.
4. Define Unimodular Transformations.
5. Define an Elliptic Functions and give an example.
6. State weierstrass $\wp$ -function.
7. What is a Legendre's relation.

✓ ***FIVE MARKS:***

1. An elliptic function without poles is a constant.
2. The sum of the residues of an elliptic function is zero.
3. A nonconstant elliptic function has equally many poles as it has zeros.

✓ ***TEN MARKS:***

1. Prove that there exists a basis (W_1, W_2) such that the ration $T = \frac{W_2}{W_1}$ satisfies the following condition:

a. Im T>0,

b. $-\frac{1}{2} < \operatorname{Re} T \leq \frac{1}{2}$,

c. $|T| \geq 1$

d. $\operatorname{Re} T \geq 0$ if $|T| = 1$.

Also show that the ratio T is uniquely determined by these consitions and there is a choice of two, four, or six corresponding bases.

2. Derive the differential equation satisfied by the weierstrass $\wp(z)$ function.
3. A discrete module consists either of zero alone, of the integral multiples nω of a single complex number ω≠0, or of all linear combinations $n_1\omega_1 + n_2\omega_2$ with integral coefficients of two numbers ω_1, ω_2 with nonreal ratio $\frac{\omega_2}{\omega_1}$.
4. Prove that any two bases of the same module are connected by a unimodular transformation.
5. The zeros $a_1, a_2, a_3, ...a_n$ and poles $b_1, b_2, b_3 ....b_n$ of an elliptic function satisfy $a_1 + ... + a_n \equiv b_1 + ... + b_n (\operatorname{mod} M)$.

6. Derive the differential equation satisfied by Weierstrass $\wp$ function.

(or)

First order differential equation of Weierstrass $\wp$ function.

7. State and prove the Weierstrass $\wp$ function.

(or)

Derive the power series of Weierstrass $\wp$ function.