St. Joseph's College of Arts and Science, Cuddalore.

Question Bank

PG Research Department of Mathematics

Class: II M.Sc Mathematics

Subject Name: FUNCTIONAL ANALYSIS

Subject Code: PMT1017

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UNIT – I

TWO MARKS:

1. Prove that the norm is a continuous function.
2. Prove that $\left| \|x\| - \|y\| \right| \leq \|x-y\|$ in a normed linear space.
3. Define the second conjugate space for a normed linear Space N.
4. Define Banach space.
5. Define normed linear space give examples.
6. Define conjugate space of N.
7. Define conjugate of an operator.
8. Define second conjugate space.
9. Define isometric isomorphism of normed linear space.
10. 10) Define normed linear space and Banach space.
11. 11) Define continuous linear transformation.
12. 12) P.T $\|T_1T_2\| \leq \|T_1\|\|T_2\|$ for $T_1, T_2 \in \mathrm{B}(\mathrm{N})$

FIVE MARKS:

1. State and prove The uniform boundedness theorem.
2. State and prove Banach-steishouse theorem.
3. If N and N′ are normal linear spaces then the set ß(N,N′) of all continuous linear transformation of N into N′ is itself a normal linear space with

Respect to the pointwise linear operation and Norm defined (i) further if N′ is a Banach space Then ß(N,N′) is also Banach space.

4. Prove that a non-empty subset X of a normed Linear space N is bounded if and only if f(X) is a Bounded set of numbers for each f in N*.

5. If P is a projection on a Banach space B and if M and N are its range and null space respectievely then P.T M & N are closed linear subspaces of B such that B=M $\oplus$ N.

6. If N is normed linear space and x_0 is a non-zero Element of N, then prove that there exist an Element f_0 ϵ N*. such that Such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.

7. Construct the closed unit spheres under the norms $\|x\|_1$, $\|x\|_2$ and $\|x\|_\infty$ in R².

TEN MARKS:

1. State and prove the Hahn-Banach theorem.

2. If B and B′ are Banach spaces and if T is a Continuous linear transformation of B onto B^{-1} then prove that the image of each open sphere centered on the origin in B contains an open sphere centred on the origin B^{-1}.

3. (i) If B and B′ are banach space and if T is a Continuous linear transformation of B and B′ then prove that T is an open mapping.

   (ii) If B is a banach space prove that the B is Reflexive if and only if B* is reflexive.

4. Define projection on Banach space and state prove the closed graph theorem.

5. Let M be closed linear subspace of a normed linear space N .If the norm of coset X+M in the quotient space N/M is defined by $\|x+M\| = \inf\{\|x+m\| : x \in M\}$ the prove N/M is a normed linear space also show that if N is a banach space so N/M.

6. State and prove open mapping theorem.

7. State and prove the closed graph theorem.

8. If M is a closed linear subspace of a normed linear space N and x_o is a vector not in M then prove that there exist a functional f_o in N* such that $f_o(M)=0$ and $f_o(x_o)\neq 0$.

9. If M is a closed linear subspace of a normal linear space N then prove that N/M is a Banach space whenever N is Banach.

10. Let M be linear subspace of normed linear space N and f be functional defined on M.If x_o is a vector not in M and if $M_o = M+[x_o]$ is the linear subspace spaned by M and x_o then prove that f can be extended to a functional defined on M_o such that $\|f_0\| = \|f\|$.

11. Let N & N′ be normed linear spaces and T be Linear transformation of N into N′ the following Conditions on T are all equivalent to one another

(i) T is continuous.

(ii) T is continuous at origin in sense that x→0 T(x_n)→0

(iii) There exist a real number $K \geq 0$ with $\|T(x)\| \leq K\|(x)\|$ for every x ϵ N.

(iv) If $S = \{x : \|x\| < 1\}$ is closed sphere in N the Image T(S) is bounded set in N′.

UNIT – II

TWO MARKS:

1. Prove that $S^{\perp}$ is a closed linear subspace of H.
2. Prove that l_2^n is a Hilbert space.
3. Define an orthonormal set in a Hilbert space.
4. Given an example for a Hilbert space.
5. Prove that the inner product in a Hilbert space is jointly continuous.
6. Establish Pythagorean Theorem.
7. If S is a non-empty subset of a Hilbert space, show that $S^{\perp} = S^{\perp\perp}$.

FIVE MARK:

1. If M and N are closed linear subspace of a Hilbert space H such that $M \perp N$, then prove that the linear sub space M+N is also closed.
2. If $\{e_1, e_2 \ldots\ldots e_n\}$ is finite orthonormal set in a Hilbert space H and x is any vector in H, then prove that $\sum_{i=1}^{n} (x, e_i)^2 \leq \|x\|^2$. Also show that $(x - \sum (x, e_i)e_i) \perp e_j$ foe each j.
3. If M is closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.
4. Prove that a Hilbert space H is separable $\Leftrightarrow$ every orthonormal set in H is countable.

TEN MARKS:

1. If B is a complete Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by

   $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x + iy\|^2$

   Then prove that B is a Hilbert space.
2. i) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

   ii) Prove that a closed convex subset C of a Hilbert space H contain a unique vector of smallest norm.
3. i) Prove that a closed convex subset C of a Hilbert space H contain a unique vector of smallest norm.

   ii) Let M be a closed linear subspace of a Hilbert space H and Let be a vector not in M and let be the distance for x to M. Then prove that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.
4. i) State and prove Schwartz Inequality.

   ii) Prove that inner product in a Hilbert space satisfies

   $x_n \rightarrow x,\ y_n \rightarrow y \Rightarrow (x_n, y_n) \rightarrow (x, y)$

   iii) Prove that Hilbert space satisfies parellogram law.
5. State and prove Bessel's Inequality.
6. Prove that every non-zero Hilbert space contain a complete orthonormal set by proving Zorn's lemma.

7. If M is a closed linear subspace of a Hilbert space H, then prove that H=M.
8. If H is a Hilbert space and $\{e_i\}$ is an orthonormal set in H then prove that the following conditions are all equivalent to one another:

   i) $\{e_i\}$ is complete.

   ii) $x \perp \{e_i\} \Rightarrow x=0$.

   iii) If x is an arbitrary vector in H, then $x=\sum (x,e_i)e_i$.

   iv)If x is an arbitrary vector in H, then $\|x\|^2 = \sum | (x,e_i) |^2$

9. i) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp$ M.

   ii) if M is a linear subspace of a Hilbert space H, Show that the set of all linear combinations of vectors in S is dense in H $\Leftrightarrow s^{\perp} = \{0\}$.

UNIT – III

TWO MARKS:

1. Prove that adjoint operation $T \to T^*$ on B(H) satisfy T**=T
2. Define self adjoint and unitary operators
3. Define normal and unitary operators
4. Prove that an operator on H is self adjoint iff $(T_{x,x})$ is real for all $x \in H$
5. Define projections
6. Define norm-preserving mapping
7. Show that $\|T^*\| = \|T\|^2$
8. Give an example of norm preserving operator

FIVE MARKS:

1. Let H be a Hilbert space and let f be a an arbitrary functional in H^*.Then there exists a unique vector y in H such that, f(x)=(x,y) for every x in H
2. If $A_1$and A_2 are self-adjoint operators on H then their product A_1A_2 is self-adjoint $\Leftrightarrow A_1A_2 = A_2A_1$
3. If T is an operator on H for which $(T_{x,x})$ = 0 for all x, then T = 0.

4. If A is a positive operator on H, then I+A is non-singular. In particular ,I+T^* T and I+TT^* are non-singular for an arbitrary operator T on H.
5. If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then $N_1 + N_2$ and $N_1 N_2$ are normal.
6. If N is normal operator on H, then $\|N^2\| = \|N\|^2$.
7. Prove that a closed linear subspace M of H is invariant under a operation T $\Leftrightarrow M^\perp$ is invariant under T^*.
8. If P and Q are the projections on closed linear subspaces M and N of H, then M $\perp$ N $\Leftrightarrow$ PQ = 0 $\Leftrightarrow$ QP =0.
9. If T is an operator on H, then prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute.
10. If P is a projection on H with range M and null space N,then M $\perp$ N $\Leftrightarrow$ Pis self-adjoint and in this case, N= $M^\perp$.

TEN MARKS:

1. (i)Let H be a Hilbert space and let f be an arbitrary functional in H^*. Then there exists a unique vector y in H such that f(x)=(x,y) for every x in H.

   (ii) If N is a normal operator on H. Prove that $\|N^2\| = \|N\|^2$.
2. Prove that the adjacent operations $T \to T^*$ on $\mathscr{B}$(H) has the following properties

   (i) $(T_1 + T_2)^*$=T_1^*+T_2^*

   (ii) $(\alpha T)^*$=$\bar{\alpha}T^*$

   (iii)($T_1 T_2)^*$=$T_2^* T_1^*$

   (iv) T**=T

   (v) $\|T^*\| = \|T\|$

   (vi) $\|T^*T\| = \|T\|^2$

3. The real banach space of all self-adjoint operators on H is a partially ordered set whose linear structure and order structure are related by the following properties
   (i) If $A_1 \le A_2$,when A_1 +A$\le$ A_2+A for every A
   (ii) If $A_1 \le A_2$ and $\alpha \ge 0$,then $\alpha A_1 \le \alpha A_2$

4. An operator T on H is normal $\Leftrightarrow$ $\|T_x^*\|$=$\|T_x\|$ for every x

5. If T is an operator on H, then T is normal $\Leftrightarrow$ its real and imaginary parts commute.

6. P.T. if T is an operator on H, then the following conditions are all equivalent to one another

(i) $TT^* = I$

(ii) (T_x, T_y) = (x,y) for all x and y

(iii) $\|T_x\| = \|x\|$ for all x

7. An operator T on H is unitary $\Leftrightarrow$ it is an isometric isomorphism of H onto itself.

8.If P and Q are the projections on closed linear subspaces M and N of H. Prove that the following statements are all equivalent to one another

(i) P≤Q

(ii) $\|P_x\| \le \|Q_x\|$ for every x

(iii) $M \le N$

(iv) PQ = P

(v) QP = P

UNIT – IV

TWO MARKS:

1. Let T be an arbitrary operator on $H, \lambda_1, \lambda_2, ..., \lambda_m$ be these eigen values and $M_1, M_2, ..., M_m$ be their corresponding eigen spaces. If T is normal, then prove that M_i's are pair wise orthogonal.
2. Define Kronecker delta.
3. What are the properties of the determinant of an operator T.
4. If T is normal, then prove that M_i reduces T.
5. What two matrices are similar?
6. Define total matrix algebra of degree n.
7. Give the spectral resolution of a normal operator.
8. Define eigenvector, eigenvalue and eigenspace.

9. What is conjugate transpose of $\lfloor \alpha_{ij} \rfloor$.

FIVE MARKS:

1. Let T be an arbitrary operator on $H, \lambda_1, \lambda_2, ..., \lambda_m$ be these eigen values and $M_1, M_2, ..., M_m$ be their corresponding eigensapces. If T is normal, then prove that $M_i's\ span\, H$.
2. If T is normal then prove that the eigen spaces of are pairewise orthogonal.
3. If T an operator on a finite dimensional Hilbert space, then prove the following statements.
   i) T is singular if and only if $0 \in \sigma(T)$.
   ii) If T is non-singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.
   iii) If A is non-singular, then $\sigma(ATA^{-1}) = \sigma(T)$.
   iv) If $\lambda \in \sigma(T)$ and if is a polynomial, then $p(\lambda) \in \sigma(p(T))$.
4. If B is a basis for H and T an operator whose matrix relative to B is $\lfloor \alpha_{ij} \rfloor$, then prove that T is non-singular $\Leftrightarrow \lfloor \alpha_{ij} \rfloor$ is non-singular and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$.
5. Prove that the dimension of $\mathrm{B}(H)$ is n^2.
6. State the spectral theorem and prove that if T is normal then $M_i's$ are pair wise orthogonal.
7. If T is an arbitrary operator on H, then prove that eigen values of T constitute a non-empty subset of the complex plane. Also prove that the number of points in tis set does not exceed the dimension n the space H.
8. If T is normal, then prove that M_i reduces T.

TEN MARKS:

1. Let T be a normal operator on H with spectrum $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ and use the spectral resolution of T to prove the following statements:
   i.) T is self-adjoint if and only if each λ_i is real;
   ii.) T is positive if and only if λ_i for each i;
   iii.) T is unitary if and only if $|\lambda_i| = 1$ for each i.
2. i) Let B be basis for H ,and T an operator whose matrix relative to B is α_{ij}. Prove that T is non-singular $\Leftrightarrow \lfloor \alpha_{ij} \rfloor$ is non-singular, and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$.
   ii) Show that the dimension of $\mathrm{B}(H)$ is n^2.

3. Prove that two matrices in A_n are similar iff they are the matrices of a single operator on H relative to different bases in H.
4. If T is normal then prove that the $M_i's\ span\ H$.
5. State and prove Spectral Theorem.
6. If T an operator on a finite dimensional Hilbert space, then prove the following statements.
   i) T is singular if and only if $0 \in \sigma(T)$.
   ii) If T is non-singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.
   iii) If A is non-singular, then $\sigma(ATA^{-1}) = \sigma(T)$.
   iv) If $\lambda \in \sigma(T)$ and if P is a polynomial, then $p(\lambda) \in \sigma(p(T))$.
7. If $B = \{e_i\}$ is a basis for H then prove that the mapping $T \to [T]$, which assigns to each operator T its matrix relative to B is an isomorphism of the algebra $\mathrm{B}(H)$ on to the total matrix algebra A_n.
8. If T is normal then prove that x is an eigen vector of T with eigen value $\lambda = x$ is an eigen vector of T^* with eigen value $\overline{\lambda}$.
9. Prove that the matrices of a single operator of different bases are similar to each other.
10. i) State the spectral theorem and prove that.
    $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ and
    ii) $T^*T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_m|^2 P_m$.

UNIT – V

TWO MARKS:

1. Define Banach Algebra.
2. Define disc Algebra.
3. Define Topological divisor of zero in a Banach Algebra
4. If A is a division algebra then prove that it equals the set of all scalar multiples of the identity.
5. What is meant by resolvent equation?

FIVE MARKS:

1. Prove that $\sigma(x)$ is non empty.
2. Prove that Z is a subset of S.
3. Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homomorphism of G onto itself, where G is the set of all regular elements in a Banach Algebra A.
4. If G is an open set then prove that S is a closed set.
5. Derive resolvent equation.
6. Prove that $\sigma(x^n) = \sigma(x)^n$

TEN MARKS:

1. Let A be a Banach Algebra and $x \in A$. Prove that $r(x) = \lim \|x^n\|^{\frac{1}{n}}$.
2. Let R be the intersection of all maximal left ideal if $r \in R$ then prove that 1-r is left regular and thereby regular.
3. Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.
4. (i) Prove that $\sigma(x)$ is non empty.

   (ii) Let S be the set of all singular elements and Z, the set of all topological divisors. Prove that the boundary of S is a subset of Z.
5. If I is a proper closed two-sided ideal in A then prove that A/I is a Banach Algebra.