ST. JOSEPH'S COLLEGE OF ARTS & SCIENCE

(AUTONOMOUS)

CUDDALORE-1

SUB: PARTIAL DIFFERENTIAL EQUATIONS
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PARTIAL DIFFERENTIAL EQUATIONS

UNIT-1

2 MARKS

1. Form the PDE eliminating arbitrary function from z = xy + f $(x^2 + y^2)$.

2. Eliminating arbitrary function from f (x + y + z, $x^2 + y^2 + z^2) = 0$

3. Form the PDE by eliminating of constant in z = $(x^2 + a)(y^2 + b)$

4. Solve: z = ax$e^y + \frac{1}{2}a^2e^{2y} + b$

5. Solve: $y^2\frac{z}{x}p + xzq = y^2$

6. When first order partial differential equation said to be compatible.

7. Find the characteristic of the equation $u_{xx} + 2u_{xy} + sin^2 x u_{yy} + u_y = 0$.

8. Find the adjoint of differential operator L(u) = $u_{xx} - u_t$.

9. Write the canonical form for elliptic equation.

10. Find the general integral of follow linear PDE Pz - qz = $z^2 + (x+y)^2$.

5 MARKS

1. Solve: $(x-a)^2 + (y-b)^2 + z^2 = a^2 + b^2$

2. Find the characteristic function of the equation pq = z and hence determine the integral surface which passes through parabola x= 0 , y^2 = z.

3. Find the system which interacts the surface of the system z(x + y) = c(3z+1) orthogonally which passes through the circle $x^2 + y^2 = 1$, z = 1.

4. Show that PDE $p^2 + q^2$ = 1 and $(p^2 + q^2)$x = pz are compatible.

5. Find the complete integral of $(p^2 + q^2)$y = qz.

6. Consider the equation $u_{xx} + x^2 u_{yy} = 0$.

7. Find $Lu = a(x)u\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u$ construct its adjoint L^*.

8. Derive the lagrangian equation of the partial differential equation of first order.

9. Classify and transform to a canonical form $sin^2 xu_{xx} + sin\ 2x.u_{xy} + cos^2 xu_{yy} = x$.

10. Show that xp – yq = x and $x^2p + q = xz$compatible and hence find a solution.

10 MARKS

1. Find the PDE of the following plane of the form where x,y,z intercepts is equal to unity.

2. Find the characteristic function of the equation pq = z and determine the integral surface which Passes through the straight line x = 1, z = y.

3. Show that xp – yq = x and $x^2p + q = xz$ compatible and hence find a solution.

4. Obtain the canonical form of the parabolic equation and hyperbolic equation.

5. Solve: $u_{xx} - 2sinx\ u_{xy} - cos^2 xu_{yy} - cosx\, u_y = 0$ to the canonical form.

6. Solve: $u_{xx} + xyu_{yy} = 0$ of the canonical form.

7. Obtain Riemann's method.

8. Obtain the Riemann solution for the equation, $\frac{\partial^2 u}{\partial x \partial y} = F(x, y)$

Given (i) u = f(x) on Γ

(ii) $\frac{\partial u}{\partial n} = g(x) on\, \Gamma$ where Γ is the curve y = x.

9. Verify that the green function for the equation $\frac{\partial^2 u}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$

Subject to u = 0, $\frac{\partial u}{\partial x} = 3x^2 \ on \ y = x$ is given by v(x,y, ε, η) = $\frac{(x+y)\{2xy+(\varepsilon-\eta)(x-y)+2\varepsilon\eta\}}{(\varepsilon+\eta)^3}$&

Obtain the solution of the form u = (x-y) ($2x^2$-xy+ $2y^2$).

10. Show that the Green's function for the equation $\frac{\partial^2 u}{\partial x \partial y}$ + u = 0 is v(x,y, ε;η) =

$J_0\sqrt{2(x-\varepsilon)(y-\eta)}$

Where J_0denotes Bessel's function of the first kind of order zero.

UNIT-2

2 MARKS:

1. Write the poisson's equation.
2. State mean value theorem for harmonic functions.
3. State maximum-minimum principle.
4. Define exterior dirichlet problem for a circle.
5. Define interior dirichlet problem for a circle.
6. Find the complete integral of p(1+q) = qz.

5 MARKS:

1. Derive the laplace equation.
2. Derive the poisson equation.
3. If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.
4. Prove that if the dirichlet problem for a bounded region has a solution, then it is unique.
5. Derive the separation of variables.

10 MARKS:

1. State and prove mean value theorem for harmonic functions.
2. Show that if the two-dimensional laplace equation $\nabla^2 u = 0$ is transformed by introducting plane polar coordinates, r, θ defined by the relations x = r cosθ, y = r sinθ, it takes the form

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{r}^2} + \frac{\mathbf{1}}{r}\frac{\partial u}{\partial r} + \frac{\mathbf{1}}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = \mathbf{0}.$$

3. Show that in cylindrical coordinates r, θ, z defined by the relations x=rcosθ , y=rsinθ, z= z, the laplace equation ∇^2u=0 takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{\mathbf{1}}{r}\frac{\partial u}{\partial r} + \frac{\mathbf{1}}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \mathbf{0}$$

4. Show that the spherical polar coordinates r, θ, φ defined by the relations x= rsinθ cosφ, y= rsinθ sinφ,

z = rcosθ, the laplace equations ∇^2u=0 takes the form,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{\mathbf{1}}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{\mathbf{1}}{sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = \mathbf{0}$$

5. Derive the dirichlet problem for a rectangle.

6. Derive the Neumann problem for a rectangle.

7. Derive exterior dirichlet problem for a circle.

8. Show that the velocity potential for an irrotational flow of an incompressible fluid satisfies the laplace solution.

9. Solve ∇^2u=0, 0$\leq x \leq a,\ 0 \leq y \leq b$Satisfying the BCs: u(0,y)=0, u(x,0)=0, u(x,b)=0

$\frac{\partial u}{\partial x}(a,y) = T sin^3 \frac{\pi y}{a}$.

10. Solve: using the method of separation of variable.
11. Derive the interior dirichlet problem for a circle.

Unit-3

2 MARKS:

1. State the insulated boundary condition.
2. What are the different types of boundary conditions?
3. State any one property of Dirac delta function.
4. State the first boundary condition of heat conduction.

5 MARKS:

1. Prove that for any continuous function $f(t)$, $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$.
2. Obtain the Fourier heat conduction equation.
3. Obtain the solution of a parabolic equation using separation of variables.
4. Derive the three possible solutions of the heat conducting equation $\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}$

10 Marks

1. In a L dimensional infinite solid $-\infty < x < \infty$ the surface $a < x < b$ is initially maintained at temperature to and at zero temperature everywhere out the surface show that,
   T(x, t) = $\frac{T_0}{2}$ [$\mathbf{erf}\frac{b-x}{\sqrt{4\alpha t}} - \mathbf{erf}\frac{a-x}{\sqrt{4\alpha t}}$] where erf is an error function.

2. Solve the 1-dimensional diffusion is the region $0 \le x \le \pi$; $t \ge 0$, show that conditions
   (i) T remains finite as t $\rightarrow \infty$.
   (ii) T = 0, if x = 0 and $\pi\ \forall\ t$.
   (iii) At t = 0, T = $\begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$.

3. A uniform rod of length L where surface is thermally insulated is initially at temperature $\theta = \theta_0$ at time t = 0 one end is suddenly coded at $\theta = 0$ subsequently maintained at this temperature the order end remains thermally insulated $\theta(x, t)$.

4. Find the solution of the one dimensional satisfying the following,
   (i) T is bounded as $t \to \infty$
   (ii) $\frac{\partial T}{\partial x} \mid_{x=0} = 0 \ \forall t$
   (iii) $\frac{\partial T}{\partial x} \mid_{x=a} = 0 \ \forall t$
   (iv) $T(x,0) = x(a-x), 0 < x < a$

5. Let H be a Hilbert space and let $f \in H^*$. Then prove that there exist a unique vector y in H such that $f(x) = (x, y)$

6.
   (i) Let P be a projection on H with range M and null space N then prove that $M \perp N$ iff P is self adjoint
   (ii) Prove that $\langle P_{x,x} \rangle \geq 0$

7. Solve $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, -\infty < x < \infty, t > 0$

UNIT-4

2 MARKS

1. Define wave function.
2. What is plane harmonic wave (or) Define monochromatic wave?
3. Write the one- dimensional wave equation.
4. What is period of wave.
5. Define frequency.
6. What is domain of dependence?
7. Write the Riemann- volterra solution
8. What are normal modes of vibration?
9. Define normal frequencies.
10. Define fundamental frequency.

11. Write the solution of non- homogenous equation.

12. Write down the Hamilton's principle.

13. Derive the D'Almbert's solution for one dimensional wave equation.

14. Write D'Almbert's solution.

5 MARKS

1. A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement u(x,0) =u_0 sin($\frac{\pi x}{L}$). The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string.

2. Obtain the solution of the wave equation $u_{tt} = c^2 u_{xx}$

under the following conditions:

i) u(0,t) = u(2,t) = 0

ii) u(x,0) = $sin^3 \frac{\pi x}{2}$

iii)$u_t(x, 0) = 0$

3. The heat condition in a thin round insulated red with heat sources present is described by the PDE, $u_t - \alpha u_{xx} = \frac{F(x,t)}{\rho}, 0 < x < \rho, t > 0$. sub to Bc's = u(0,t) = u(ρ, t) $= 0$

Ic's = u(x,0) = $\rho(x), 0 \leq x \leq \rho$

Where ρ& c are constraints and F is continuous function of x and t. Find u (x,t).

4. Find a particular solution of the problem described by

PDE: $y_{tt} - c^2 y_{xx}$ = g(x) cos wt, 0 < x <L ,t> 0

Bc's: y (0,t) = y (L,t) =0, t > 0

Where g(x) is a piecewise smooth function and w is a positive constant.

5. Solve the equation$\frac{\partial T}{\partial t} = \frac{\partial^1 T}{\partial x^1}$satisfying the condition

i) T=0 when x=0 and x=1

ii) $T=\begin{cases} 2x & 0\le x\le 1/2 \\ 2(1-x) & 1/2\le x\le 1 \end{cases}$; when t=0

6. Solve: $\frac{\partial\theta}{\partial t} = \frac{\partial^1\theta}{\partial x^1}$, $0\leq x \leq a, t > 0$

sub to conditions $\theta(0,t) = \theta(a,t) = 0$ and $\theta(x,\theta) = \theta_0$ (constant)

7. Derive the D'Alembert's solution of one dimensional wave equation.

8. The forces x=0, x=a of a finite slab are maintained at zero temperature. The initial distribution of temperature in the slab is given by$T(x,0) = f(x),\ 0 \leq x < a$. Determine the temperature at subsequent times.

9. S.T the equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ satisfying the condition

i) T → 0 as t → ∞

ii) T = 0 for x= 0 and x = a, t>0.

iii) T = x when t = 0 and 0 < x <a is T(x,t) = $\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\left(\frac{n\pi}{a}\right)x\rho^{-(\frac{n\pi}{a})^2 t}$

10. A conducting bar of uniform cross section lies along the x-axis with its ends, at x=0 & x=l. The lateral surface is insulated. There are no heat sources within the body. The ends are also insulated. The initial temperature is lx - x^2, $0 \leq x \leq l$. Find the temperature distribution in the bar for t > 0.

10 MARKS

1. Derive the solution of one dimensional wave equation by canonical reduction.

2. Derive D'Alembert's solution by the initial value problem.

3. Derive variables separable solution by vibrating string.

4. Prove that the total energy of a string which is fixed at the points x= 0 , x = L and executing small transverse vibrations, is given by $\frac{1}{2}\ T\ \int_0^L[(\frac{\partial y}{\partial x})^2 + \frac{1}{c^2}(\frac{\partial y}{\partial t})^2]$ dx where $c^2 = \frac{T}{\rho}$, ρ is the

uniform linear density and T is the tension. Show also that if y= f(x-ct), $0 \leq x \leq L$, then the energy of the wave is equally divided between potential energy and kinetic energy.

5. Solve $u_{tt} = c^2 u_{xx} + F(x,t),\ 0 \leq x \leq L, t = 0$ satisfying the condition.

6. Derive the solution of non- homogeneous equation by forced vibrations.

7. Explain boundary and initial value problem for two dimensional wave equation method of eigen function.

Unit-5

2 Marks:

1. What is half-range series
2. Write down the wave equation in terms of cylindrical co-ordination
3. Define the Hankel Function $H_0^{(1)}(x)$.
4. Define the HankelFunctions involved in deriving the periodic solution of one-dimensional wave equation in cylindrical co-ordinates.

5 MARKS:

1. Find the particular solution of the problem described by,
PDE: $y_{tt} - c^2 y_{xx}$ = g(x) coswt , 0< x < t , t> 0
BCS: y(0,t) = y(L,t) = 0 , t> 0 ,where S(x) is a piece wise force of W is a positive constant.

2. Solve the IVP described by,
PDE: $u_{tt} - c^2 u_{xx}$ = F(x,t) , $-\infty < x < \infty$, $t \geq 0$ with the data
(i) F(x,t) = 4x + t , (ii) u(x ,0) =0 (iii) u_t(x ,0) = cos hbx
(ii)
3. Derive the wave equation representing the transverse vibration of string in the form ,
$\frac{\partial^2 u}{\partial t^2} = e^2\{1 + (\frac{\partial u}{\partial x})^2\}^2 \frac{\partial^2 u}{\partial x^2}$.

10 MARKS:

1. A uniform string of line density ρ is stricted to tension P C^2 and exceed a small transverse vibration in a plane through the undistributed line of string the end x=0, L of string are fixed the string is at rest with the point x=b drown through a small distance and released at time t=0 find an expression for the displacement y(x, t).

2. Periodic solution of the one dimensional wave equation in cylindrical co-ordinates.