St. Joseph's College of Arts and Science, Cuddalore.

Question Bank

PG & Research Department of Mathematics

Class: I M.Sc Mathematics

Subject Name: Measure Theory

Subject Code: PMT807

Staff Name: Mr. A. Virgin

II Semester question bank

UNIT - I

1. Prove that the interval (a,∞) is measurable.
2. Define an outer measure. Prove that the outer measure of an interval is its length.
3. (i) State and prove Little Wood's first principle.
   (ii) Let E be a given set. Then prove that the following 5 statements are equivalent:
   i) E is measurable.
   ii) Given Є>0,∃ an open set O⊃E with m^*(OE)<Є.
   iii) Given Є>0,∃ a closed set F⊂E with $m^*(EF) < Є$
   iv) There is a G in G_δ with E⊂G,m^*(GE)=0
   v) There is a F in F_σ with F⊂E,$m^*(EF) = 0$
4. Prove: Let E be a measurable set of finite measure, and <f_n> a sequence of measurable functions defined on E. Let f be a real valued function such that for each x Є E we have $f_n(x) \to f(x)$. Then given Є > 0 and δ>0 , there is a measure set ACE with mE< δ and an integer N such that for all x$\in A$ and all n ≥ N,$|f_n(x) - f(x)|$< Є.
5. (i) If <E_n> is an finite decreasing sequence of measurable sets with $E_{n+1} \subset E_n$ for each n and $mE_1 < \infty$ then prove that $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$
6. State Littlewood's three principles. State and prove Egoroff's theorem.
7. Construct a non measurable set.
8. Prove that the collection M of measurable sets is a$\sigma -$ algebra.

Unit – II

1. State and prove Fatou's lemma.
2. State and prove the Bounded Convergence theorem
3. State and prove Monotone convergence theorem
4. Lebesgue convergence theorem
5. If f is a non negative function which is integrable over a set E then prove that for given Є > 0 there is a δ > 0 such that for every set A ⊂ E with mA < δ we have $\int_A f < \epsilon$

   If f is a nonnegative function which is integrable over a set E, then prove that for given Є>0 there is a δ>0 such that for every set A⊂E with mA<δ we have $\int_A f < \epsilon$

   Let f be a non negative function which is integrable over a set E. Then given Є >0 there is a δ >0 such that for every set A⊂E with mA<δ , we have $\int_A f < \epsilon$. Prove.
6. If f and g are bounded measurable functions defined on a set E of finite measure, then prove that $\int_E af + bf = a\int_E f + b\int_E g$and if f=g a.e then $\int_E f = \int_E g$
7. State and prove the necessary and sufficient condition that f be measurable.
   State and prove the necessary and sufficient condition that f be measurable.
8. Define the convergence in measure. If <f_n> is a sequence of measurable functions that converges in measure to f, prove that there is a subsequence <f_{nk}> that converges to f almost everywhere.
9. Let f be defined and bounded on a measurable set E with mE finite. In order that $\inf_{f\le\psi} \int_E \psi(x)dx = \sup_{f\ge\varphi} \int_E \varphi(x)dx$ for all simple functions $\varphi and \psi$, it is necessary and sufficient that f be measurable. Prove.

Unit – III

1. Prove that a function f is of bounded variation on [a,b] iff f is the difference of two monotone real-valued functions on [a,b].
2. State and prove Vitali's Lemma.
3. If f is a absolutely continuous on [a,b] and $f'(x) = 0$ almost everywhere then prove that f is constant.
4. State and prove Jensen Inequality.
5. If f is bounded and measurable on [a,b] and $F(x) = \int_a^x f(t)dt + F(a)$then prove that$F'$(x)=f(x)for almost all x is [a,b]
6. Let f be an increasing real-valued function on [a,b]. Prove that f is differentiable almost everywhere, the derivative is measurable and $\int_a^b f(x)dx \le f(b) - f(a)$.
7. If f is integrable on [a,b] and $\int_a^b f(t)dt = f$ for all x [a,b]. Prove that f(t)=0 almost everywhere in [a,b].
8. A function F is an indefinite integral if and only if it is absolutely continuous. Prove.
9. If f is absolutely continuous on [a,b] and a.e., prove that f is constant

Unit – IV

1. State and prove Hahn Decomposition Theorem.

2. State and prove Lebesgue Convergence Theorem on general measure.
3. State and prove Radon-Nikodym theorem.
4. Let (x,$\mathcal{B}$) be a measurable space, <μ_n> a sequence of measures that converge setwise to a measure μ and <f_n> a sequence of non-negative measurable functions that converge point-wise to the set function f. Prove that $\int f d\mu \leq \underline{\lim} \int f_n d\mu_n$
5. If for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for, then prove that there is a unique measurable extended real-valued function f on X such that $f \leq \alpha on B_\alpha and f \geq \alpha on X \sim B_\alpha$
6. If for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha \sim B_\beta) = 0$ for α<β , then prove that there is ameasurable function f such that $f \leq \alpha$ almost everywhere on $B_\alpha$and $f \geq \alpha$ almost everywhere on $X \sim B_\alpha$ Also prove the uniqueness.
7. If E is a measurable set such that 0<$\gamma E < \infty$ then prove that there is a positive set A contained in E with $\gamma A > 0$
8. If $E_i \in \mathcal{B}, \mu E_1 < \infty$ and $E_i \supset E_{i+1}$, prove that $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu E_n$

Unit – V

1. State and prove caratheodory theorem.
2. Prove that the class $\mathcal{B}$of μ^*- measurable sets is aσ −algebra. If is $\bar{\mu} is \mu^*$restricted to $\mathcal{B}$, then $\bar{\mu}$is a complete measure on$\mathcal{B}$
3. Prove that the class $\mathcal{B}$ of μ^* measurable sets is a α -algebra.
4. If E is a set in $\mathcal{R}_{\sigma\delta}$ with $\mu_x \gamma(E) < \infty$, then prove that the function defined by is a measurable function of x and g(x)= γE_x is a measurable function of x and $\int g d\mu = \mu_x \gamma(E)$
5. If x is a point of X and E a set in $\mathcal{R}_{\sigma\delta}$, then prove that $E_x$is a measurable subset of Y.
6. If A∈$\mathcal{R}$, then prove that A is measurable with respect to μ^*
7. If {($A_i X B_i$)} is a countable disjoint collection of measurable rectangles whose union is a measurable rectangle AXB, then prove that λ(AXB)=Σλ(A_iXB_i).
8. State and prove Fubini's Theorem.
9. State and prove Tonelli theorem.
10. Let μ be a measure on an algebra and μ^*be the outer measurable induced by μ. Let E be any set. Prove that for ∈>0 , there is a set $A \in \mathcal{A}_\sigma$ with E⊂A and $\mu^* A \leq \mu^* E + \epsilon$. There is also a set $B \in \mathcal{A}_{\sigma\delta}$ with E⊂B and $\mu^* E = \mu^* B$
11. Let μ be a measure on an algebra A and μ^* be the outer measure induced by μ. Let E be any set. Prove that for ∈>0 , there is a set A∈$\mathcal{A}_\sigma$ with E⊂ A and μ^*A≤ $\mu^* E + \epsilon$. There is also a set B∈$\mathcal{A}_{\sigma\delta}$ with E⊂B and $\mu^* E = \mu^* B$.